is constructed in a piecewise smooth manner on the natural cell decompositions of $T_{pl}(S,V)$ and $T(S-V)\times\mathbb{R}^V_{>0}$. These cell decompositions are derived from the Delaunay triangulations of the underlying spaces by the work of Penner and Rivin. In the second step, we exam the restriction of discrete curvature map on the $DC([d])$. By step 1, $DC([d])$ is naturally a Euclidean space. Using a variational principle developed by Luo in 2004, we show that the discrete curvature map on $DC([d])/\mathbb{R}_{>0}$ is the gradient of a strictly convex function. Thus, it is an injective map. On the other hand, by analyzing the degeneration of discrete conformality of triangles and using a result of Akiyoshi, we show that the image $K(DC([d]))$ is closed in $Y=(-\infty,2\pi)^V\cap\{x\in\mathbb{R}^V\,|\,\sum_v x(v)=2\pi\chi(S)\}$. Since both $DC([d])/\mathbb{R}_{>0}$ and Y are connected manifolds of the same dimension, we conclude that $K|$ is a homeomorphism and thus prove theorem 2.

There are several open problems related to the discrete conformality. First, we do not know how to prove theorem 2 for non-compact surfaces. Second, we conjecture that discrete conformality converges to the classical conformality when triangulations become finer and finer. The numerical evidences to this conjecture are very strong. However, a rigorous proof is still lacking.

References

[1] D. Gu, F. Luo, J. Sun, T. Wu, *A discrete uniformization theorem for polyhedral surfaces*, arXiv:1309.4175.

Problem Session

Problems compiled by Daniele Alessandrini, session chaired by Norbert A'Campo

Problem 1. (Norbert A'Campo) Given an immersed curve in $\mathbb{R}^2$ with transverse self-intersection, we can count the number of self-intersections. Denote by $N(g)$ the number of curves as above with exactly g self-intersections, up to isotopy. Give an interpretation to the power series:

$$\sum_{g\geq 0} N(g)z^g$$

This problem is related with matrix models and to the problem of counting cells in $M_{g,1}$, the moduli space of Riemann surfaces with one marked point.

Problem 2. (Athanase Papadopoulos) Understand the relation between the Galois group and the Teichmüller space. The absolute Galois group, $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ acts on the set of isotopy classes of finite planar trees. This is almost in bijection with the set of complex polynomials in 1 variable having 0 and 1 as their only critical values, given such a polynomial p the planar tree is given by $p^{-1}([0,1])$. The coefficients of such polynomials lie in some number field, hence the absolute Galois group acts on the set of polynomial by changing the coefficients. This action is faithful. The problem is to understand the orbits at the level of planar trees. This is in relation with problem 1.

Problem 3. (Ursula Hamenstädt) A Hurwitz curve is a Riemann surface with maximal symmetry, for example the one obtained from the triangle surface with angles $\pi/2, \pi/3, \pi/7$. The Galois group acts on the collection of Hurwitz curves. Robert Kucharczyk proved that this action is faithful. What is the smallest genus of a Riemann surface such that the action sends it to one that is not isomorphic?

Problem 4. (Lizhen Ji)

(1) Does the Teichmüller space $T_{g,n}$ admit a complete $CAT(0)$-metric invariant under the mapping class group MCG?
(2) Does the outer space O_n admit a complete $CAT(0)$-metric invariant under $\mathrm{Out}(F_n)$?

Note that $T_{g,n}/MCG$ and $O_n/\mathrm{Out}(F_n)$ are not compact, so the questions don't imply that the groups MCG and $\mathrm{Out}(F_n)$ are $CAT(0)$, which is false.

Problem 5. (Lizhen Ji) Consider the Teichmüller space $T_{g,n}$ with the Teichmüller metric. The moduli space $M_{g,n} = T_{g,n}/MCG$ has finite volume. How special is this volume? Is it algebraic or transcendental? One may conjecture that it is transcendental, and some special value of an L-function.

Similar question for the first eigenvalue of the Laplacian for the Teichmüller metric, $\lambda_1(T_{g,n})$. McMullen proved that it is strictly positive, but how special is it?

In Thurston's list of open problems he asks how special are the volumes of closed hyperbolic 3-manifolds.

Problem 6. (Lizhen Ji) What is the number of integral points on the moduli space?

Problem 7. (Sumio Yamada) Thurston's stretch map between two hyperbolic surfaces minimises the Lipschitz constant of maps in its homotopy class. Is there a distance on the target surface that makes the stretch map harmonic? This question is motivated by the analogy with the Teichmüller map, minimising the quasiconformal constant of maps in its homotopy class. This map is harmonic with reference to the flat metric with singularities induced by the Teichmüller quadratic differential.

Problem 8. (Michael Wolf) The Hitchin component $\mathrm{Hit}(n)$ is a connected component of the variety of representations of the fundamental group of a closed surface in $\mathrm{PSL}(n, \mathbb{R})$ up to conjugation. Hitchin proved that given a fixed Riemann surface structure X on the surface, $\mathrm{Hit}(n)$ is parametrised by the space $\bigoplus_{i=2}^{n} H^0(X, K^i)$, where K is the canonical bundle of X. Labourie asked the question whether it is possible to parametrise $\mathrm{Hit}(n)$ with the space $\bigcup_{X\in T_g} \bigoplus_{i=3}^{n} H^0(X, K^i)$. Labourie and Loftin proved this for $n = 3$ using affine spheres and the Monge-Ampere equation.

Problem 9. (Hiroshige Shiga) Let X be a finite type Riemann surface with negative Euler characteristic, and consider a holomorphic map $\varphi : X \to M_g$. This map can be lifted to a holomorphic map $\phi : \mathbb{H}^2 \to T_g$, with monodromy

$\rho : \pi_1(X) \to \mathrm{Mod}_g$. A result of rigidity by Imayoshi and Shiga says that given two such holomorphic maps φ_1, φ_2, if their monodromies agree, then $\varphi_1 = \varphi_2$.

Is this again true if we consider the representation ρ' that is the composition of the monodromy ρ with the representation $\mathrm{Mod}_g \to \mathrm{Sp}(2g, \mathbb{Z})$? More explicitely, if two holomorphic maps φ_1, φ_2 satisfy $\rho'_1 = \rho'_2$, is it true that $\varphi_1 = \varphi_2$?

If this is true, we can have an effective bound on the number of holomorphic maps $X \to M_g$.

Problem 10. (Ursula Hamenstädt) What is the smallest genus of a closed surface that can be mapped to M_g in such a way that the associated monodromy is injective?

Problem 11. (Mustafa Korkmaz)

(1) Is the mapping class group linear?
(2) Let S be a surface with genus $g \geq 3$ and boundary components $\delta_1, \ldots, \delta_n$. Denote by τ_γ the Dehn twist around γ. Is it possible to write $\tau_{\delta_1} \cdots \tau_{\delta_n}$ as a product of positive Dehn twists about non-separating simple closed curves? This is true for $n \leq 4g + 4$, but what about the general case?
(3) If $g \geq 3$ and $n \geq 3$, is it possible to write $\tau_{\delta_1} \cdots \tau_{\delta_n}$ as a product of an arbitrarily large number of positive Dehn twists about non-separating simple closed curves? The motivation comes from the theory of Stein filling contact 3-manifolds.
(4) Can we understand $H_3(\mathrm{Mod}_g, \mathbb{Z})$? For the moment we can only understand $H_3(\mathrm{Mod}_g, \mathbb{Q})$.
(5) Is the Torelli group finitely presented?
(6) Is there a finite index subgroup of Mod_g that contains the Torelli group?

Problem 12. (Gregor Masbaum) The group $\mathrm{Mod}_{g,n}$ acts on Teichmüller space and on the curve complex. Consider the normal subgroup $t(k)$ that is generated by all the k-th powers of Dehn twists. Is it possible to find some nice space with a good action of $\mathrm{Mod}_{g,n}/t(k)$? This is interesting for the theory of quantum representations, where we can construct representations with kernel containing $t(k)$. This problem can be solved for $\mathrm{Mod}_{1,1}$, is it possible to generalise? (Leonid O. Chekhov provided additional insight about the properties of $\mathrm{Mod}_{1,1}$ and its relations with Kontsevich's matrix models, asking if this can also be generalised).

Problem 13. (Jørgen E. Andersen) Let $M = \mathrm{Hom}(\pi_1(S), \mathrm{SU}(2))/\mathrm{SU}(2)$ or $M = \mathrm{Hom}(\pi_1(S), \mathrm{SL}(2, \mathbb{C}))//\mathrm{SL}(2, \mathbb{C})$, and let M' be the subset corresponding to irreducible representations. The mapping class group Γ acts on M and M'. Let U be one of the rings $\mathcal{O}(M), L^2(M), C_c^\infty(M), C^\infty(M')$. What is $H^1(\Gamma, U)$? The motivation for this question is to understand if there is a unique *-product on the set of functions that are invariants under Γ.

Problem 14. (Nariya Kawazumi) Let S be a compact connected oriented surface. Let $\hat{\pi}(S)$ denote the set of homotopy classes of curves in S, and $\mathbb{R}\hat{\pi}(S)$ denote the Goldman Lie algebra. A theorem states that if S is closed, then the center of $\mathbb{R}\hat{\pi}(S)$ is the subspace generated by 1. What happens for surfaces with boundary?

Is it true that the center is the linear span of 1 and of all the powers of boundary curves?

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